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OUTPUT–FEEDBACK SLIDING-MODE CONTROL WITH GENERALIZED SLIDING SURFACE FOR CIVIL STRUCTURES UNDER EARTHQUAKE EXCITATION

E. E. MATHEU[1], M. P. SINGH[1]* AND C. BEATTIE[2]

[1] *Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.*
[2] *Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.*

SUMMARY

The paper presents a control scheme based on the sliding-mode-control approach. The analytical formulation focuses on the development of (1) a convenient, systematic and general scheme to achieve the so-called regular form of the equations of motion required to uncouple the control actions from the sliding motion description, (2) a systematic treatment of control redundancy where the number of sliding constraints imposed are less than the number of independent control actions, and (3) a method to improve sliding surface design by incorporating auxiliary dynamic systems. Both full-state-feedback and output-feedback cases are considered. In the output-feedback formulation, a generalized procedure is developed so that arbitrary combinations of unavailable system states (unmeasured displacements or velocities, for example) need not participate in the design of the sliding surface or the controller. A controller design utilizing only bounding information on the intensity of ground motion and the unmeasured states is proposed. The analytical formulation developed herein is applied to a 10-storey building structure to obtain the numerical results. The advantages of introducing auxiliary systems in the design of the sliding surface and the corresponding controller are noted. The results for both full-state-feedback and output-feedback cases are presented and compared to demonstrate applicability of the proposed control scheme. © 1998 John Wiley & Sons, Ltd.

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1. INTRODUCTION

An active structural control system consists of three basic components: (1) sensing devices which measure the input and output response of the structural system; (2) information processing and decision-making systems which determine the timing and magnitude of the required control actions, and; (3) actuators which apply the necessary control actions. This paper deals with the second component and it focuses on the development of a suitable control algorithm for seismic-response control of civil engineering structures.

Several control algorithms or schemes have been used in the past for structural-control applications. One of the most common of all the proposed schemes is the linear quadratic regulator approach or its modifications. Another approach that has been gaining acceptance because of its versatility for application to linear and non-linear systems, and also its superb robustness under parametric and input uncertainties, is the sliding-mode-control approach. Yang and his colleagues brought this approach to the attention of the civil engineering community through several enlightening papers[1–6] on its application in control of civil structures. In this context, the papers by the writers[7–10] on this subject have also advanced the state-of-the-art of utilizing this approach for active, semi-active and hybrid control strategies.

* Correspondence to: M. P. Singh, 305 Norns Hall, Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A. E-mail: mpsingh@vt.edu

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Perhaps, the most crucial step in the implementation of a sliding-mode-control scheme is the design of the sliding surface. The sliding surface is a set of constraints imposed on the system by appropriate control actions. The sliding surface could involve static constraints, or more generalized constraints with a dynamic description. Inappropriate selection of these constraints can lead to an improper design, where increased control effort does not necessarily translate into better control of the response. The selection of a 'proper' sliding surface or constraints is thus quite important in achieving an efficient and flexible control design. The selection of a sliding surface is affected by the number of independent control actions that can be applied, by the choice of the optimality criterion used in the design process and by the number and type of the response quantities that are available for feedback to the controller. The number of applied control actions determines how many independent constraints can be imposed; of course, the number of constraints cannot be more than the number of control actions. If there are more controls than constraints then the system has control redundancy, and it requires special treatment in the development of the algorithm. Since not all states are amenable for measurements, this practical consideration must also be considered in the design of a sliding surface. This paper deals with some of these issues, focusing on the specific aspects of seismic-response control of civil structures.

A chosen sliding surface should be such that the corresponding constrained motion (called as the sliding motion) is always stable and has some desirable characteristics. To facilitate the design process, it is first necessary to uncouple the constrained motion description and the constraining control actions, such that these control actions do not appear explicitly in the equations of the sliding motion. This can be achieved by transformation of the state equations into their regular form. Sometimes this form can be achieved by a simple reordering of the equations of motion. In a more general case, however, it may require a special transformation to achieve these objectives. This paper utilizes the QR decomposition to effect such a transformation.

The approach described here also employs a dynamic description of the sliding surface, where the constraints are defined on the outputs of two auxiliary dynamical systems, and not directly in terms of the state variables as is done in the classical sliding-mode-control approach. This provides additional flexibility for further turning of the design to meet different performance specifications. This design flexibility becomes important if the active control system is a part of an integrated protection system in which the response reduction responsibilities are distributed among structural members, passive devices and active systems.

Formulation for both full-state-feedback and output-feedback situations are considered in the development of the sliding surface. A general approach is formulated to eliminate the explicit effect of the unmeasured states on the design of sliding surface. This approach permits the utilization of different combinations of measured and unmeasured states. To verify the applicability of the formulation for various cases, several sets of numerical results are obtained for a 10-storey building structure. The ground motions recoded in four different earthquake events are used to evaluate the effectiveness of the control schemes for varied disturbances.

In the design of a control system there are several other important issues one must examine, e.g. time delay effects, actuator saturation and malfunction, and other reliability concerns. These have not been addressed in this study.

2. SYSTEM EQUATIONS

2.1. Equations of motion

The equations of motion of a linear building system with n_f degrees of freedom under seismic excitation can be written as follows:

$$\mathbf{M}\ddot{\mathbf{d}} + \mathbf{C}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = -\,\mathbf{M}\mathbf{r}\ddot{x}_g + \mathbf{D}\mathbf{u} \tag{1}$$

The n_f-dimensional vector $\mathbf{d}$ describes the displacement configuration of the system. The displacement vector $\mathbf{d}$ could be defined in terms of displacements relative to ground, or for a commonly used model of a shear

building, in terms of interstorey drifts. The vector **r** represents the influence of the ground excitation $\ddot{x}_g(t)$ on each degree of freedom. The $n_f \times n_f$ matrices **M**, **C** and **K** represent the mass, damping and stiffness matrices, respectively. The vector **u** contains the m_c active control actions whose locations are identified through the $n_f \times m_c$ location matrix **D** having $\mathrm{rank}(\mathbf{D}) = m_c \leqslant n_f$.

The equations of motion (1) can be rewritten in state space form as

$$\dot{\boldsymbol{\eta}} = \mathbf{A}\boldsymbol{\eta} + \mathbf{B}\mathbf{u} + \mathbf{e}\ddot{x}_g \tag{2}$$

where the n $(=2n_f)$ dimensional state vector $\boldsymbol{\eta}$ and the state matrix **A** are given, respectively, by

$$\boldsymbol{\eta} = \begin{Bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{Bmatrix} \quad \text{and } \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n_f} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \tag{3}$$

and the control input matrix **B** and disturbance input vector **e** are given, respectively, by

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \quad \text{and } \mathbf{e} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{r} \end{bmatrix} \tag{4}$$

From the point of view of practical feasibility of implementation, it is necessary to consider the situation in which the measurements on all the state variables may not be available for control purposes. Usually, only a subset of the physical variables, such as displacements and velocities, can be directly obtained. It is desirable, therefore, to design a controller based on the availability of such partial information.

Let the n_a-dimensional vector $\boldsymbol{\eta}_a$ indicate those state variables that can be directly measured. By using $n \times n$ permutation matrix **S**, it is possible to define

$$\bar{\boldsymbol{\eta}} = \begin{Bmatrix} \boldsymbol{\eta}_u \\ \boldsymbol{\eta}_a \end{Bmatrix} = \mathbf{S}\boldsymbol{\eta} \tag{5}$$

where the last n_a components of the vector $\bar{\boldsymbol{\eta}}$ correspond to the available variables $\boldsymbol{\eta}_a$, and the remaining $n_u = n - n_a$ unmeasured or unavailable variables are collected in the vector $\boldsymbol{\eta}_u$. Equation (5) can be written in partitioned form as follows:

$$\begin{Bmatrix} \boldsymbol{\eta}_u \\ \boldsymbol{\eta}_a \end{Bmatrix} = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{21} & \mathbf{S}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \dot{\mathbf{d}} \end{Bmatrix} \tag{6}$$

where now the partitions of the matrix **S** have appropriate dimensions and directly identify the participations of the co-ordinates **d** and $\dot{\mathbf{d}}$ in the definitions of $\boldsymbol{\eta}_a$ and $\boldsymbol{\eta}_u$.

2.2. Regular form

The strategy for the design of a sliding-mode controller consists of two basic steps. In the first step, the system is required to satisfy a set of m_s constraints among the state variables. These imposed constraints define a surface in the state space, known as the *sliding surface*.[11–13] The second step consists of determining the control actions which either keep the system on the sliding surface or ensure that it remains in the neighbourhood of such a surface. The imposition of m_s constraints on the system will require at least an equal number of control actions. When the system is forced to satisfy the m_s constraints imposed by the sliding surface, the resulting motion is referred to as *sliding motion*.[11–13]

The goal of this control approach is to design a sliding surface such that the sliding motion is stable and has some desirable characteristics. Since the control actions are only applied to impose the sliding constraints, it proves to be convenient to uncouple the sliding motion from the control actions. That is, it is convenient to obtain a system representation for which the sliding motion can be described without explicit control action terms. Such a representation is known as the regular form.[11–13] In the following section, a new formulation is presented that achieves such a form systematically.

For any matrix $\mathbf{Z}$, let $\mathscr{R}(\mathbf{Z})$ denote the range of $\mathbf{Z}$, i.e. the column space of the matrix $\mathbf{Z}$. Let $\mathbf{V}$ be an $n_f \times m_c$ matrix whose columns constitute an orthonormal basis for $\mathscr{R}(\mathbf{M}^{-1}\mathbf{D})$. Also, let $\mathbf{U}_1$ be another matrix of dimension $n_f \times (n_f - m_c)$, the columns of which constitute an orthonormal basis for $\{\mathscr{R}(\mathbf{M}^{-1}\mathbf{D})\}^{\perp}$, i.e. the orthogonal complement of $\mathscr{R}(\mathbf{M}^{-1}\mathbf{D})$. These two matrices can be obtained by first applying Gram–Schmidt process to the columns of $\mathbf{M}^{-1}\mathbf{D}$ and then completing an orthonormal basis for $\mathfrak{R}^{n_f}$. Furthermore, let an $n_f \times m_c$ matrix $\mathbf{U}_2$ be defined as

$$\mathbf{U}_2 = \mathbf{V}\mathbf{Q} \tag{7}$$

where $\mathbf{Q}$ is an orthogonal matrix obtained from the QR decomposition of the product $[\mathbf{S}_{12}\mathbf{V}]^{\mathrm{T}}$ as follows:

$$[\mathbf{S}_{12}\mathbf{V}]^{\mathrm{T}} = \mathbf{Q}\mathbf{R} \tag{8}$$

where $\mathbf{S}_{12}$ was defined in equation (6). The matrix $\mathbf{R}$ has dimensions $m_c \times n_u$ with an upper triangular structure which depends on the rank of $[\mathbf{S}_{12}\mathbf{V}]^{\mathrm{T}}$. Both matrices $\mathbf{Q}$ and $\mathbf{R}$ depend upon the availability of information about the state variables. For the extreme case in which all states are available, i.e. $\boldsymbol{\eta}_a = \boldsymbol{\eta}$, $\mathbf{S}_{12}$ becomes a matrix with *no rows* and $\mathbf{R}$ becomes a matrix with *no columns*. In this case, $\mathbf{Q}$ can be chosen arbitrarily and it is assigned as $\mathbf{Q} = \mathbf{I}_{m_c}$.

In terms of the matrices $\mathbf{U}_1$ and $\mathbf{U}_2$, an $n \times n$ matrix $\mathbf{T}$ is defined as

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_{n_f} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \tag{9}$$

Since $[\mathbf{U}_1 \;\; \mathbf{U}_2]$ is unitary by construction, it follows that $\mathbf{T}$ is unitary, i.e. $\mathbf{T}^{-1} = \mathbf{T}^{\mathrm{T}}$. This matrix is now used to define a state transformation as follows:

$$\boldsymbol{\eta} = \mathbf{T}\mathbf{y} \tag{10}$$

After substituting equation (10) into the state equations (2) and premultiplying by $\mathbf{T}^{\mathrm{T}}$, the transformed state equations are given by

$$\dot{\mathbf{y}} = \bar{\mathbf{A}}\mathbf{y} + \bar{\mathbf{B}}\mathbf{u} + \bar{\mathbf{e}}\ddot{x}_g \tag{11}$$

where

$$\bar{\mathbf{A}} = \mathbf{T}^{\mathrm{T}}\mathbf{A}\mathbf{T}, \qquad \bar{\mathbf{B}} = \mathbf{T}^{\mathrm{T}}\mathbf{B} \quad \text{and} \quad \bar{\mathbf{e}} = \mathbf{T}^{\mathrm{T}}\mathbf{e} \tag{12}$$

By considering the definition of the matrix $\mathbf{T}$ given by equation (9), it is immediately clear that the transformed matrix $\bar{\mathbf{B}}$ takes the following form:

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \hat{\mathbf{B}} \end{bmatrix} \tag{13}$$

in which $\hat{\mathbf{B}}$ is an $m_c \times m_c$ non-singular matrix given by

$$\hat{\mathbf{B}} = \mathbf{U}_2^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{D} \tag{14}$$

With this structure for the input matrix $\bar{\mathbf{B}}$, the first $n - m_c$ equations of (11) do not contain any control terms now. When $m_c = m_s$, i.e. when the number of control actions equals the number of sliding constraints, this representation leads to a set of sliding motion equations, given by the upper $n - m_c$ equations of (11), without explicit control terms. Thus, in this case, equations (11) represents the regular form of the state equations. However, in the case of control redundancy, when $m_c > m_s$, additional constraints will have to be imposed on the admissible control actions to obtain sliding motion equations uncoupled from the control terms. This is discussed in the following sections.

3. SLIDING SURFACE AND SLIDING MOTION

As mentioned before, the sliding surface represents a set of suitably chosen m_s constraints. Let the transformed state vector $\mathbf{y}$ defined in equation (10) be accordingly partitioned as

$$\mathbf{y} = \begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{Bmatrix} \tag{15}$$

so that the first $n_r = n - m_s$ components are arranged in a vector $\mathbf{y}_1$ and the remaining m_s variables are collected in a vector $\mathbf{y}_2$.

Normally, the sliding surface is defined as a set of linear constraints on the state vector $\mathbf{y}$. However, to add flexibility in the choice of desirable sliding motions, here the sliding constraints are defined in terms of two derived m_s-dimensional vectors $\tilde{\mathbf{y}}_1$ and $\tilde{\mathbf{y}}_2$ as follows:

$$\mathbf{s} = \tilde{\mathbf{y}}_1 + \tilde{\mathbf{y}}_2 = \mathbf{0} \tag{16}$$

The vectors $\tilde{\mathbf{y}}_1$ and $\tilde{\mathbf{y}}_2$ are the outputs of two auxiliary dynamical systems with the following state-space realizations of order n_{s_1} and n_{s_2}, respectively:

$$\begin{cases} \dot{\boldsymbol{\varphi}}_1 = \bar{\mathbf{F}}_1\boldsymbol{\varphi}_1 + \bar{\mathbf{G}}_1\mathbf{y}_1 \\ \tilde{\mathbf{y}}_1 = \bar{\mathbf{H}}_1\boldsymbol{\varphi}_1 + \bar{\mathbf{C}}_{s1}\mathbf{y}_1 \end{cases} \quad \text{and} \quad \begin{cases} \dot{\boldsymbol{\varphi}}_2 = \bar{\mathbf{F}}_2\boldsymbol{\varphi}_2 + \bar{\mathbf{G}}_2\mathbf{y}_2 \\ \tilde{\mathbf{y}}_2 = \bar{\mathbf{H}}_2\boldsymbol{\varphi}_2 + \bar{\mathbf{C}}_{s2}\mathbf{y}_2 \end{cases} \tag{17}$$

The $m_s \times m_s$ matrix $\bar{\mathbf{C}}_{s2}$ is always assumed to be non-singular, i.e. $\text{rank}(\bar{\mathbf{C}}_{s2}) = m_s$. The matrices $\bar{\mathbf{F}}_1$, $\bar{\mathbf{G}}_1$, $\bar{\mathbf{F}}_2$ and $\bar{\mathbf{G}}_2$ are preselected and they act as design parameters, whereas the matrices $\bar{\mathbf{H}}_1$, $\bar{\mathbf{C}}_{s1}$, $\bar{\mathbf{H}}_2$ and $\bar{\mathbf{C}}_{s2}$ will be determined based on some optimality condition imposed on the sliding motion. It is also required that the number of constraints should not be more than the number of independent control actions. That is, the control redundancy index, defined as $m_r = m_c - m_s$, satisfies $m_r \geqslant 0$.

The auxiliary systems used to define the sliding surface can be incorporated into the state equations to define an augmented system of order $n_e = n_{s_1} + n_{s_2} + n$ as follows:

$$\dot{\mathbf{y}}_e = \bar{\mathbf{A}}_e\mathbf{y}_e + \bar{\mathbf{B}}_e\mathbf{u} + \bar{\mathbf{e}}_e\ddot{x}_g \tag{18}$$

in which

$$\bar{\mathbf{A}}_e = \begin{bmatrix} \bar{\mathbf{F}}_1 & \mathbf{0} & \bar{\mathbf{G}}_1 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{F}}_2 & \mathbf{0} & \bar{\mathbf{G}}_2 \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix}, \quad \mathbf{y}_e = \begin{Bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \end{Bmatrix} \quad \text{and} \quad \bar{\mathbf{e}}_e = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \bar{\mathbf{e}}_1 \\ \bar{\mathbf{e}}_2 \end{Bmatrix} \tag{19}$$

where $\bar{\mathbf{A}}$, $\bar{\mathbf{B}}$ and $\bar{\mathbf{e}}$ have been partitioned to conform with the partitioning of $\mathbf{y}$. The sliding surface can also be written more compactly in terms of the augmented state vector $\mathbf{y}_e$ as follows:

$$\mathbf{s} = \bar{\mathbf{C}}_e\mathbf{y}_e = \mathbf{0} \tag{20}$$

where the matrix $\bar{\mathbf{C}}_e$ is defined as

$$\bar{\mathbf{C}}_e = [\bar{\mathbf{H}}_1 \;\; \bar{\mathbf{H}}_2 \;\; \bar{\mathbf{C}}_{s1} \;\; \bar{\mathbf{C}}_{s2}] \tag{21}$$

To obtain the control actions $\hat{\mathbf{u}}$ which will force the system to stay on the sliding surface, the following condition is used:

$$\dot{\mathbf{s}}|_{\mathbf{u}=\hat{\mathbf{u}}} = \mathbf{0} \tag{22}$$

By utilizing the state equations (18) and the sliding surface equation (20) in the sliding constraint (22), we obtain the following equation to define $\hat{\mathbf{u}}$:

$$\bar{\mathbf{C}}_e\bar{\mathbf{B}}_e\hat{\mathbf{u}} = -\bar{\mathbf{C}}_e\bar{\mathbf{A}}_e\mathbf{y}_e - \bar{\mathbf{C}}_e\bar{\mathbf{e}}_e\ddot{x}_g \tag{23}$$

Based on the definitions of the matrices $\bar{\mathbf{C}}_e$ and $\bar{\mathbf{B}}_e$, it can be shown that the rank of the coefficient matrix $\bar{\mathbf{C}}_e\bar{\mathbf{B}}_e$ in equation (23) is equal to m_s. When $m_c = m_s$, there is a redundancy of control actions and a unique solution $\hat{\mathbf{u}}$ cannot be determined unless further restrictions are imposed on the admissible controls. It can be shown that if $\hat{\mathbf{u}}$ is required to satisfy

$$\bar{\mathbf{B}}_1\hat{\mathbf{u}} = \mathbf{0} \tag{24}$$

then the following unique solution can be obtained for (23):

$$\hat{\mathbf{u}} = -[\bar{\mathbf{C}}_e\bar{\mathbf{B}}_e]^{\ddagger}\bar{\mathbf{C}}_e\{\bar{\mathbf{A}}_e\mathbf{y}_e + \bar{\mathbf{e}}_e\ddot{x}_g\} \tag{25}$$

in which the matrix $[\bar{\mathbf{C}}_e\bar{\mathbf{B}}_e]^{\ddagger}$ denotes a particular right inverse of $[\bar{\mathbf{C}}_e\bar{\mathbf{B}}_e]$, given by

$$[\bar{\mathbf{C}}_e\bar{\mathbf{B}}_e]^{\ddagger} = \hat{\mathbf{B}}^{-1}\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{C}}_{s2}^{-1}\end{bmatrix} \tag{26}$$

The development of equation (25) from equations (23) and (24) requires some algebraic steps which are not presented here to conserve the space. By substituting equation (25) into equation (18), it can be shown that the behaviour of the system subjected to the control $\hat{\mathbf{u}}$ can be described by the following system of order n_c, with $n_c = n_{s_1} + n_{s_2} + n_r$:

$$\dot{\mathbf{y}}_c = \bar{\mathbf{A}}_c\mathbf{y}_c + \bar{\mathbf{B}}_c\mathbf{y}_2 + \bar{\mathbf{e}}_c\ddot{x}_g \tag{27}$$

where

$$\bar{\mathbf{A}}_c = \begin{bmatrix}\bar{\mathbf{F}}_1 & \mathbf{0} & \bar{\mathbf{G}}_1\\ \mathbf{0} & \bar{\mathbf{F}}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{11}\end{bmatrix}, \quad \bar{\mathbf{B}}_c = \begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{G}}_2\\ \bar{\mathbf{A}}_{12}\end{bmatrix}, \quad \mathbf{y}_c = \begin{Bmatrix}\boldsymbol{\varphi}_1\\ \boldsymbol{\varphi}_2\\ \mathbf{y}_1\end{Bmatrix} \quad \text{and} \quad \bar{\mathbf{e}}_c = \begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\\ \bar{\mathbf{e}}_1\end{Bmatrix} \tag{28}$$

Note that, with the imposition of the condition (24), equations (27) have no control term appearing explicitly.

Finally, to obtain the sliding motion equations the static constraint $\mathbf{s} = \mathbf{0}$ must be incorporated into equation (27). Considering equation (21), and noting that $\bar{\mathbf{C}}_{s2}$ is non-singular, the sliding surface constraint (20) can be solved for $\mathbf{y}_2$ as

$$\mathbf{y}_2 = \bar{\mathbf{K}}_c\mathbf{y}_c \tag{29}$$

in which

$$\bar{\mathbf{K}}_c = -\bar{\mathbf{C}}_{s2}^{-1}[\bar{\mathbf{H}}_1 \;\; \bar{\mathbf{H}}_2 \;\; \bar{\mathbf{C}}_{s1}] \tag{30}$$

Substituting equation (29) into equation (27), we obtain finally

$$\dot{\mathbf{y}}_c = [\bar{\mathbf{A}}_c + \bar{\mathbf{B}}_c\bar{\mathbf{K}}_c]\,\mathbf{y}_c + \bar{\mathbf{e}}_c\ddot{x}_g \tag{31}$$

This reduced system of equations describe the behaviour of the system under ideal sliding motion conditions. The matrices $\bar{\mathbf{A}}_c$, $\bar{\mathbf{B}}_c$ and $\bar{\mathbf{K}}_c$, and therefore the characteristics of this constrained motion, depend on the two sets of matrices ($\bar{\mathbf{F}}_1$, $\bar{\mathbf{G}}_1$, $\bar{\mathbf{H}}_1$, $\bar{\mathbf{C}}_{s1}$) and ($\bar{\mathbf{F}}_2$, $\bar{\mathbf{G}}_2$, $\bar{\mathbf{H}}_2$, $\bar{\mathbf{C}}_{s2}$) defining the sliding surface.

4. SLIDING SURFACE DESIGN

The sliding surface design problem involves the calculation of matrices $\bar{\mathbf{H}}_1$, $\bar{\mathbf{C}}_{s1}$, $\bar{\mathbf{H}}_2$ and $\bar{\mathbf{C}}_{s2}$. For this, it is common to use the minimization of a performance index or cost function which is quadratic in the state vector components $\mathbf{y}_1$ and $\mathbf{y}_2$. In the following section, however, a performance index is defined in terms of two sets of output variables related to the auxiliary systems (17) instead of $\mathbf{y}_1$ and $\mathbf{y}_2$. First the formulation for

the full-state-feedback case, in which measurements are available for all states, is presented. Since it may not be feasible to obtain direct information for all the state variables, the formulation for the output-feedback case, which depends on measurements of only a few states, is also presented later.

4.1. Full-state feedback

The performance index chosen to be minimized is of the following form:

$$J = \int_{t_h}^{\infty} (\mathbf{q}_1^{\mathrm{T}}\mathbf{Q}_1\mathbf{q}_1 + \mathbf{q}_2^{\mathrm{T}}\mathbf{Q}_2\mathbf{q}_2)\,\mathrm{d}t \tag{32}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are $n_q \times n_q$ and $m_s \times m_s$ positive semi-definite and positive definite weighting matrices, respectively. The vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ have dimensions n_q and m_s, respectively, and are defined by the following output equations:

$$\mathbf{q}_1 = \bar{\mathbf{D}}_1\boldsymbol{\varphi}_1 + \bar{\mathbf{E}}_1\mathbf{y}_1 \quad \text{and} \quad \mathbf{q}_2 = \bar{\mathbf{D}}_2\boldsymbol{\varphi}_2 + \bar{\mathbf{E}}_2\mathbf{y}_2 \tag{33}$$

in which it is required that the $m_s \times m_s$ matrix $\bar{\mathbf{E}}_2$ be non-singular. The variables $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ are the state variables corresponding to the realizations indicated in equations (17). The lower limit of the integral in equation (32) is explicitly denoted as t_h to indicate that this performance index is evaluated for the system under sliding conditions. This time t_h (hitting time) is the time in which the system reaches the sliding surface, and without any loss of generality we can set $t_h = 0$.

It can be shown that this type of cost function is equivalent to a frequency-dependent weighting of the variables $\mathbf{y}_1$ and $\mathbf{y}_2$ in the performance index.[14] This provides a convenient way for selectively penalizing some frequencies more than others in order to achieve some desirable system characteristics. The introduction of the matrices $\bar{\mathbf{D}}_1$, $\bar{\mathbf{E}}_1$, $\bar{\mathbf{D}}_2$ and $\bar{\mathbf{E}}_2$ in the definition of the outputs $\mathbf{q}_1$ and $\mathbf{q}_2$ also offers additional flexibility for further tuning of the design to meet performance specifications.

Substituting $\mathbf{q}_1$ and $\mathbf{q}_2$ from equations (33) into equation (32), the performance index can be expressed as

$$J = \int_{0}^{\infty} (\mathbf{y}_c^{\mathrm{T}}\bar{\mathbf{W}}_{11}\mathbf{y}_c + 2\mathbf{y}_c^{\mathrm{T}}\bar{\mathbf{W}}_{12}\mathbf{y}_2 + \mathbf{y}_2^{\mathrm{T}}\bar{\mathbf{W}}_{22}\mathbf{y}_2)\,\mathrm{d}t \tag{34}$$

where $\mathbf{y}_c$ was defined in equation (28) and the weighting matrices are given by

$$\bar{\mathbf{W}}_{11} = \begin{bmatrix} \bar{\mathbf{D}}_1^{\mathrm{T}}\mathbf{Q}_1\bar{\mathbf{D}}_1 & \mathbf{0} & \bar{\mathbf{D}}_1^{\mathrm{T}}\mathbf{Q}_1\bar{\mathbf{E}}_1 \\ \mathbf{0} & \bar{\mathbf{D}}_2^{\mathrm{T}}\mathbf{Q}_2\bar{\mathbf{D}}_2 & \mathbf{0} \\ [\bar{\mathbf{D}}_1^{\mathrm{T}}\mathbf{Q}_1\bar{\mathbf{E}}_1]^{\mathrm{T}} & \mathbf{0} & \bar{\mathbf{E}}_1^{\mathrm{T}}\mathbf{Q}_1\bar{\mathbf{E}}_1 \end{bmatrix} \tag{35}$$

$$\bar{\mathbf{W}}_{12} = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{D}}_2^{\mathrm{T}}\mathbf{Q}_2\bar{\mathbf{E}}_2 \\ \mathbf{0} \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{W}}_{22} = [\bar{\mathbf{E}}_2^{\mathrm{T}}\mathbf{Q}_2\bar{\mathbf{E}}_2] \tag{36}$$

It is desired to minimize this cost function under the condition that the system is forced to satisfy the sliding constraint $\dot{\mathbf{s}}|_{\mathbf{u}=\hat{\mathbf{u}}} = \mathbf{0}$. The corresponding state equations are given by equation (27), and if the external excitation is omitted, they take the following form:

$$\dot{\mathbf{y}}_c = \bar{\mathbf{A}}_c\mathbf{y}_c + \bar{\mathbf{B}}_c\mathbf{y}_2 \tag{37}$$

The problem of minimizing equation (34) subject to equation (37) can be solved as a classical optimal control problem with cross-terms in the cost function. In view of the definition of $\bar{\mathbf{W}}_{12}$ in equation (36), it is noted that the cross terms in equation (34) depend only on $\boldsymbol{\varphi}_2$ and $\mathbf{y}_2$. Therefore, they depend entirely on the definition of the second output variable in equation (33). In the case of a static definition of the form $\mathbf{q}_2 = \bar{\mathbf{E}}_2\mathbf{y}_2$, it is seen that the cross terms entirely vanish from the expression of the performance index.

The optimal $\mathbf{y}_2$ is obtained following the well-known procedures as,[15]

$$\mathbf{y}_2 = -\bar{\mathbf{W}}_{22}^{-1}[\hat{\mathbf{B}}_c^T\mathbf{P} - \bar{\mathbf{W}}_{12}^T]\mathbf{y}_c \tag{38}$$

in which the $n_c \times n_c$ matrix $\mathbf{P}$ is the solution of the following Riccati equation:

$$\mathbf{P}\bar{\mathbf{A}}_c + \bar{\mathbf{A}}_c^T\mathbf{P} - \mathbf{P}\bar{\mathbf{B}}_c\bar{\mathbf{W}}_{22}^{-1}\bar{\mathbf{B}}_c^T\mathbf{P} = -\hat{\mathbf{W}}_{11} \tag{39}$$

where the matrix $\hat{\mathbf{W}}_{11}$ is given by

$$\hat{\mathbf{W}}_{11} = \bar{\mathbf{W}}_{11} - \bar{\mathbf{W}}_{12}\bar{\mathbf{W}}_{22}^{-1}\bar{\mathbf{W}}_{12}^T \tag{40}$$

Equation (38) defines a linear relation that must be satisfied by the state variables under the sliding conditions. Comparing equations (29) and (38), it follows that

$$\bar{\mathbf{K}}_c = -\bar{\mathbf{W}}_{22}^{-1}[\bar{\mathbf{B}}_c^T\mathbf{P} - \bar{\mathbf{W}}_{12}^T] \tag{41}$$

By selecting $\bar{\mathbf{C}}_{s2} = \bar{\mathbf{E}}_2^T\mathbf{Q}_2\bar{\mathbf{E}}_2$, the remaining matrices defining the sliding surface can be obtained as appropriate partitions of the matrix $\bar{\mathbf{B}}_c^T\mathbf{P} - \bar{\mathbf{W}}_{12}^T$ of the form

$$\bar{\mathbf{B}}_c^T\mathbf{P} - \bar{\mathbf{W}}_{12}^T = [\bar{\mathbf{H}}_1 \;\; \bar{\mathbf{H}}_2 \;\; \bar{\mathbf{C}}_{s1}] \tag{42}$$

Since there were no restrictions imposed on the structure of the matrix $\bar{\mathbf{C}}_e$ defining the sliding surface, the resulting control system will have, in general, full-state-feedback characteristics. The computation of $\mathbf{s}$, which depends on the auxiliary states $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ and the transformed states $\mathbf{y}_1$ and $\mathbf{y}_2$, will require information about all displacement and velocity variables in the model.

4.2. *Output feedback*

In this case, it is assumed that only n_a state variables $\boldsymbol{\eta}_a$ are available from direct measurement. The goal of the design process, distinct from that of the full-state-feedback case, is now to eliminate the contribution of the unmeasured variables $\boldsymbol{\eta}_u$ from the dynamics of the auxiliary systems (17).

Using the definitions (5) and (10), the state vector $\mathbf{y}$ can be expressed in terms of the rearranged measured and unmeasured quantities as follows:

$$\mathbf{y} = [\mathbf{ST}]^T\bar{\boldsymbol{\eta}} \tag{43}$$

and this equation can be rewritten in partitioned form as

$$\begin{Bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\eta}_u \\ \boldsymbol{\eta}_a \end{Bmatrix} \tag{44}$$

where the blocks $\mathbf{P}_{ij}$ indicate submatrices of $[\mathbf{ST}]^T$ with appropriate dimensions.

It can be shown that if the following condition is satisfied:

$$\operatorname{rank}(\mathbf{S}_{12}\mathbf{V}) \leqslant m_r \tag{45}$$

the matrix $\mathbf{P}_{21}$ in equation (44) is identically zero. The vector $\mathbf{y}_2$ can then be written as

$$\mathbf{y}_2 = \mathbf{P}_{22}\boldsymbol{\eta}_a \tag{46}$$

Thus, if condition (45) holds, the component $\mathbf{y}_2$ of the transformed state vector is not affected by unmeasured quantities.

To eliminate the contribution of $\boldsymbol{\eta}_u$ from the terms involving $\mathbf{y}_1$ in the definition of the sliding surface, let a matrix $\bar{\mathbf{M}}_1$ be defined such that its row space is contained in the null space of $\mathbf{P}_{11}^T$, here denoted by $\mathcal{N}(\mathbf{P}_{11}^T)$. That is,

$$\bar{\mathbf{M}}_1\mathbf{P}_{11} = \mathbf{0} \tag{47}$$

Premultiplying the first n_r equations of (44) by $\bar{\mathbf{M}}_1$ and considering equation (47), one can write

$$\bar{\mathbf{M}}_1 \mathbf{y}_1 = \bar{\mathbf{M}}_1 \mathbf{P}_{12} \boldsymbol{\eta}_a \tag{48}$$

Hence, the unmeasured variables can be eliminated from the definition of the first auxiliary system in equation (17) provided that the matrices $\bar{\mathbf{G}}_1$ and $\bar{\mathbf{C}}_{s1}$ have the form:

$$\bar{\mathbf{G}}_1 = \hat{\mathbf{G}}_1 \bar{\mathbf{M}}_1 \quad \text{and} \quad \bar{\mathbf{C}}_{s1} = \hat{\mathbf{C}}_{s1} \bar{\mathbf{M}}_1 \tag{49}$$

for arbitrary choices of $\hat{\mathbf{G}}_1$ and $\hat{\mathbf{C}}_{s1}$.

Therefore, if the condition (45) is satisfied and the matrices $\bar{\mathbf{G}}_1$ and $\bar{\mathbf{C}}_{s1}$ are selected as in equation (49), with $\bar{\mathbf{M}}_1$ chosen to satisfy equation (47), the unmeasured quantities $\boldsymbol{\eta}_u$ will not participate in the sliding surface definition. To incorporate these constraints in the sliding surface design an n_ζ-dimensional vector $\boldsymbol{\zeta}_1$ is defined as follows:

$$\boldsymbol{\zeta}_1 = \bar{\mathbf{M}}_1 \mathbf{y}_1 \tag{50}$$

in which the $n_\zeta \times n_r$ matrix $\bar{\mathbf{M}}_1$ is not only selected such that it satisfies the condition (47), but also such that its row space is contained in $\mathscr{R}(\mathbf{P}_{12})$. In particular, the rows of $\bar{\mathbf{M}}_1$ are chosen such that they constitute an orthonormal basis for the subspace $\mathscr{N}(\mathbf{P}_{11}^{\mathrm{T}}) \cap \mathscr{R}(\mathbf{P}_{12})$. The dimension n_ζ of the vector $\boldsymbol{\zeta}_1$ is therefore is equal to the dimension of this intersection subspace. It can be shown that if $\mathbf{P}_{21} = \mathbf{0}$, then $\mathscr{R}(\mathbf{P}_{12}) \subseteq \mathscr{N}(\mathbf{P}_{11}^{\mathrm{T}})$ and therefore this particular selection of the matrix $\bar{\mathbf{M}}_1$ is always possible.

The vector $\boldsymbol{\zeta}_1$ is used to redefine the auxiliary system governing the output variable $\mathbf{q}_1$ in the performance index as follows:

$$\begin{cases} \dot{\boldsymbol{\varphi}}_1 = \bar{\mathbf{F}}_1 \boldsymbol{\varphi}_1 + \hat{\mathbf{G}}_1 \boldsymbol{\zeta}_1 \\ \mathbf{q}_1 = \bar{\mathbf{D}}_1 \boldsymbol{\varphi}_1 + \hat{\mathbf{E}}_1 \boldsymbol{\zeta}_1 \end{cases} \tag{51}$$

Using $\mathbf{q}_1$ defined by equation (51) and $\mathbf{q}_2$ defined by equation (33) in the performance index definition (32), the cost function can be expressed as before as

$$J = \int_0^\infty (\mathbf{y}_c^{\mathrm{T}} \bar{\mathbf{W}}_{11} \mathbf{y}_c + 2\mathbf{y}_c^{\mathrm{T}} \bar{\mathbf{W}}_{12} \mathbf{y}_2 + \mathbf{y}_2^{\mathrm{T}} \bar{\mathbf{W}}_{22} \mathbf{y}_2)\, \mathrm{d}t \tag{52}$$

where the vector $\mathbf{y}_c$ and the matrices $\bar{\mathbf{W}}_{12}$ and $\bar{\mathbf{W}}_{22}$ are still the same as in equations (28) and (36), respectively, but the weighting matrix $\bar{\mathbf{W}}_{11}$ is redefined as

$$\bar{\mathbf{W}}_{11} = \begin{bmatrix} \bar{\mathbf{D}}_1^{\mathrm{T}} \mathbf{Q}_1 \bar{\mathbf{D}}_1 & \mathbf{0} & \bar{\mathbf{D}}_1^{\mathrm{T}} \mathbf{Q}_1 \hat{\mathbf{E}}_1 \bar{\mathbf{M}}_1 \\ \mathbf{0} & \bar{\mathbf{D}}_2^{\mathrm{T}} \mathbf{Q}_2 \bar{\mathbf{D}}_2 & \mathbf{0} \\ [\bar{\mathbf{D}}_1^{\mathrm{T}} \mathbf{Q}_1 \hat{\mathbf{E}}_1 \bar{\mathbf{M}}_1]^{\mathrm{T}} & \mathbf{0} & \bar{\mathbf{M}}_1^{\mathrm{T}} \hat{\mathbf{E}}_1^{\mathrm{T}} \mathbf{Q}_1 \hat{\mathbf{E}}_1 \bar{\mathbf{M}}_1 \end{bmatrix} \tag{53}$$

As seen in the previous case, the minimization of equation (52) subject to the sliding equations (37) will provide an optimal solution of the form $\mathbf{y}_2 = \bar{\mathbf{K}}_c \mathbf{y}_c$ where $\mathbf{y}_2$ depends not only on the states $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ but also, in general, on all the components of the vector $\mathbf{y}_1$. But to eliminate the effect of the unmeasured states, it is desired to obtain an optimal solution that depends only on $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$ and $\boldsymbol{\zeta}_1$ in the following form:

$$\mathbf{y}_2 = \mathbf{N}_c \begin{Bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\zeta}_1 \end{Bmatrix} \tag{54}$$

where $\mathbf{N}_c$, is to be determined. Equation (54) can be written as

$$\mathbf{y}_2 = \mathbf{N}_c \begin{Bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\zeta}_1 \end{Bmatrix} = \mathbf{N}_c \begin{bmatrix} \mathbf{I}_{n_{s1}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n_{s2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{M}}_1 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \mathbf{y}_1 \end{Bmatrix} = \mathbf{N}_c \mathbf{C}_c \mathbf{y}_c \tag{55}$$

The sliding surface design process can be cast now as an optimal output feedback problem, where the matrix $\mathbf{C}_c$ plays the role of an output matrix and $\mathbf{N}_c$ is the output feedback matrix to be determined by minimizing the cost function (52) subject to equation (37).

To eliminate the presence of cross terms in the cost function the standard procedure can be used by defining[15]

$$\mathbf{y}_2 = \hat{\mathbf{y}}_2 - \bar{\mathbf{W}}_{22}^{-1}\bar{\mathbf{W}}_{12}^{\mathrm{T}}\mathbf{y}_c \tag{56}$$

With this substitution, the cost function takes the form

$$J = \int_0^\infty (\mathbf{y}_c^{\mathrm{T}}\bar{\mathbf{W}}_{11}\mathbf{y}_c + \mathbf{y}_2^{\mathrm{T}}\bar{\mathbf{W}}_{22}\mathbf{y}_2)\,\mathrm{d}t \tag{57}$$

where the matrix $\hat{\mathbf{W}}_{11}$ has the same definition as in the full-state-feedback case, given by equation (40), but with the matrix $\bar{\mathbf{W}}_{11}$ redefined in equation (53). The corresponding state equations, under the condition $\dot{\mathbf{s}}|_{\mathbf{u}=\hat{\mathbf{u}}} = \mathbf{0}$ and absence of the external excitation term, are obtained by considering equations (37) and (56) as follows:

$$\dot{\mathbf{y}}_c = [\bar{\mathbf{A}}_c - \bar{\mathbf{B}}_c\bar{\mathbf{W}}_{22}^{-1}\bar{\mathbf{W}}_{12}^{\mathrm{T}}]\,\mathbf{y}_c + \bar{\mathbf{B}}_c\hat{\mathbf{y}}_2 \tag{58}$$

The idea is to minimize equation (57) subject to equation (58) with a solution of the form

$$\hat{\mathbf{y}}_2 = \hat{\mathbf{N}}_c\mathbf{C}_c\mathbf{y}_c \tag{59}$$

where $\hat{\mathbf{N}}_c$ is a matrix still to be determined. It is important to mention that in a general case, the transformation (56) cannot be employed to generate an output feedback solution for $\mathbf{y}_2$, because it will require complete knowledge of $\mathbf{y}_c$ which is not available. However, for this particular case, the term involving $\mathbf{y}_c$ is given by

$$\bar{\mathbf{W}}_{22}^{-1}\bar{\mathbf{W}}_{12}^{\mathrm{T}}\mathbf{y}_c = \bar{\mathbf{E}}_2^{-1}\bar{\mathbf{D}}_2\boldsymbol{\varphi}_2 \tag{60}$$

which depends entirely on the accessible states $\boldsymbol{\varphi}_2$. This allows the use of equation (56) to determine the solution $\hat{\mathbf{N}}_c$. In the sequel, necessary conditions for an optimal matrix $\hat{\mathbf{N}}_c$ are briefly derived following the approach proposed by Mendel.[16] These necessary conditions are well known in the context of optimal output-feedback-control theory. While this brief derivation is not meant to be an original contribution of this work, it is presented here, in the context of sliding mode control, for the sake of completeness and continuity of the presentation.

The closed-loop dynamics under sliding motion can be obtained by substituting $\hat{\mathbf{y}}_2$, with the structure indicated by equation (59), into equation (58) as follows:

$$\dot{\mathbf{y}}_c = [\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c]\,\mathbf{y}_c \tag{61}$$

where

$$\hat{\mathbf{A}}_c = \bar{\mathbf{A}}_c - \bar{\mathbf{B}}_c\bar{\mathbf{W}}_{22}^{-1}\bar{\mathbf{W}}_{12}^{\mathrm{T}} \tag{62}$$

These equations describe the behaviour of the system after the time instant t_h (hitting time) at which the system reaches the sliding surface. The initial conditions for equation (61) are thus given by the state of the system at this time instant, denoted by $\mathbf{y}_c(t_h) = \mathbf{y}_h$.

If the resulting closed-loop matrix $\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c$ is asymptotically stable, then it can be shown that the performance index can be evaluated in terms of the initial conditions as[17]

$$J = \mathbf{y}_h^{\mathrm{T}}\mathbf{M}_c\mathbf{y}_h \tag{63}$$

where the $n_c \times n_c$ matrix $\mathbf{M}_c$ satisfies the following condition:

$$\mathscr{G}(\tilde{\mathbf{N}}_c, \mathbf{M}_c) = \mathbf{0} \tag{64}$$

in which

$$\mathscr{G}(\hat{\mathbf{N}}_c, \mathbf{M}_c) = [\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c]^{\mathrm{T}}\mathbf{M}_c + \mathbf{M}_c[\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c] + \hat{\mathbf{W}}_{11} + \mathbf{C}_c^{\mathrm{T}}\hat{\mathbf{N}}_c^{\mathrm{T}}\bar{\mathbf{W}}_{22}\hat{\mathbf{N}}_c\mathbf{C}_c \tag{65}$$

This allows the problem to be cast in the form of a static-constrained optimization problem, but this optimization depends on the hitting state, $\mathbf{y}_h$. To remove the dependence on a specific hitting state, it is assumed that $\mathbf{y}_h$ is a random variable with zero-mean and covariance matrix $\mathbf{Y}_h = E\{\mathbf{y}_h\mathbf{y}_h^{\mathrm{T}}\}$. For an asymptotically stable closed-loop matrix $\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c$, an average performance index considering all possible hitting conditions can be redefined as follows:

$$\bar{J} = \mathrm{E}\{J\} = \mathrm{tr}\{\mathbf{M}_c\mathbf{Y}_h\} \tag{66}$$

To solve the problem of minimizing (66) subject to the constraints (64), the augmented cost $\bar{J}_a$ is defined as

$$\bar{J}_a(\hat{\mathbf{N}}_c, \mathbf{M}_c, \mathbf{\Lambda}_c) = \mathrm{tr}\{\mathbf{M}_c\mathbf{Y}_h + \mathbf{\Lambda}_c\mathscr{G}(\hat{\mathbf{N}}_c, \mathbf{M}_c)\} \tag{67}$$

in which the constraint equations have been incorporated through a Lagrange multiplier matrix $\mathbf{\Lambda}_c$. The necessary conditions for an optimal solution are obtained by setting the partial derivatives of $\bar{J}_a$ with respect to $\hat{\mathbf{N}}_c$, $\mathbf{M}_c$ and $\mathbf{\Lambda}_c$ equal to zero. This leads to the following system of coupled non-linear matrix equations:

$$\bar{\mathbf{B}}_c^{\mathrm{T}}\mathbf{M}_c\mathbf{\Lambda}_c\mathbf{C}_c^{\mathrm{T}} + \bar{\mathbf{W}}_{22}\hat{\mathbf{N}}_c\mathbf{C}_c\mathbf{\Lambda}_c\mathbf{C}_c^{\mathrm{T}} = \mathbf{0} \tag{68}$$

$$[\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c]\mathbf{\Lambda}_c + \mathbf{\Lambda}_c[\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c]^{\mathrm{T}} = -\mathbf{Y}_h \tag{69}$$

$$[\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c]^{\mathrm{T}}\mathbf{M}_c + \mathbf{M}_c[\hat{\mathbf{A}}_c + \bar{\mathbf{B}}_c\hat{\mathbf{N}}_c\mathbf{C}_c] + \mathbf{C}_c^{\mathrm{T}}\hat{\mathbf{N}}_c^{\mathrm{T}}\bar{\mathbf{W}}_{22}\hat{\mathbf{N}}_c\mathbf{C}_c = -\hat{\mathbf{W}}_{11} \tag{70}$$

These conditions were first derived by Levine and Athans,[18] but not in the context of sliding-mode control. Several different algorithms have been proposed to solve these equations using gradient techniques, successive substitution procedures and homotopy methods. A modified successive substitution, which converges to the final solution in just a few iterations, is used in the numerical examples presented later. The details of this method will be presented elsewhere.

Once an optimal solution $\hat{\mathbf{N}}_c$ is obtained and by selecting $\bar{\mathbf{C}}_{s2} = \mathbf{I}_{m_s}$, the remaining matrices defining the sliding surface can be determined as follows:

$$\bar{\mathbf{H}}_1 = -\hat{\mathbf{N}}_1, \qquad \bar{\mathbf{H}}_2 = -\hat{\mathbf{N}}_2 - \bar{\mathbf{E}}_2^{-1}\bar{\mathbf{D}}_2 \quad \text{and} \quad \bar{\mathbf{C}}_{s1} = -\hat{\mathbf{N}}_3, \tag{71}$$

where $\hat{\mathbf{N}}_1$, $\hat{\mathbf{N}}_2$ and $\hat{\mathbf{N}}_3$ are appropriate partitions of the matrix $\hat{\mathbf{N}}_c$ of the form

$$\hat{\mathbf{N}}_c = [\hat{\mathbf{N}}_1 \;\; \hat{\mathbf{N}}_2 \;\; \hat{\mathbf{N}}_3] \tag{72}$$

Finally, the sliding surface can be written in terms of the available states as follows:

$$\mathbf{s} = \mathbf{C}_e\boldsymbol{\eta}_e = \mathbf{0} \tag{73}$$

where

$$\mathbf{C}_e = [\bar{\mathbf{H}}_1 \;\; \bar{\mathbf{H}}_2 \;\; \bar{\mathbf{C}}_{s1}\mathbf{P}_{12} + \bar{\mathbf{C}}_{s2}\mathbf{P}_{22}] \quad \text{and} \quad \boldsymbol{\eta}_e = \begin{Bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\eta}_a \end{Bmatrix} \tag{74}$$

This definition for the sliding surface holds also for the full-state-feedback case, i.e. $\boldsymbol{\eta}_a = \boldsymbol{\eta}$. For this case, the matrix $\mathbf{S}$ in equation (43) becomes an identity matrix and the partitions $\mathbf{P}_{11}$ and $\mathbf{P}_{21}$ disappear from equation (44) because $n_u = 0$.

5. CONTROLLER DESIGN

The controller must attract the system state toward the sliding surface $\mathbf{s} = \mathbf{0}$ and must keep it there. A general approach based on Lyapunov's direct method is used for the design of the controller. Let V be a Lyapunov function candidate of the form

$$V = \tfrac{1}{2}\mathbf{s}^{\mathrm{T}}\mathbf{s} \tag{75}$$

The time derivative of this function is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) = \mathbf{s}^{\mathrm{T}}\dot{\mathbf{s}} \tag{76}$$

In order to assure the existence of sliding motion and to guarantee that any motion is going to be attracted to the sliding surface, the purpose of the controller is to force equation (76) to be a negative definite function. Considering the augmented state equations (18) and the definition of the sliding surface given by equation (20), it follows that

$$\dot{\mathbf{s}} = \bar{\mathbf{C}}_{\mathrm{e}}\{\bar{\mathbf{A}}_{\mathrm{e}}\mathbf{y}_{\mathrm{e}} + \bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}}\} + \bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{B}}_{\mathrm{e}}\mathbf{u} \tag{77}$$

Substituting equation (77) into equation (76), the time derivative of V takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) = \mathbf{s}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{e}}\{\bar{\mathbf{A}}_{\mathrm{e}}\mathbf{y}_{\mathrm{e}} + \bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}}\} + \mathbf{s}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{B}}_{\mathrm{e}}\mathbf{u} \tag{78}$$

Taking into account the definitions of the augmented state matrix and the sliding surface matrix, given by equations (19) and (21), and after some rearrangement, equation (78) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) = \mathbf{s}^{\mathrm{T}}\{\mathbf{K}_1\boldsymbol{\varphi}_1 + \mathbf{K}_2\boldsymbol{\varphi}_2 + \mathbf{Q}_1\mathbf{y}_1 + \mathbf{Q}_2\mathbf{y}_2\} + \mathbf{s}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}} + \mathbf{s}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{B}}_{\mathrm{e}}\mathbf{u} \tag{79}$$

where the following notation was introduced:

$$\begin{aligned} \mathbf{K}_1 &= \bar{\mathbf{H}}_1\bar{\mathbf{F}}_1, \qquad \mathbf{Q}_1 = \bar{\mathbf{H}}_1\bar{\mathbf{G}}_1 + \bar{\mathbf{C}}_{\mathrm{s}1}\bar{\mathrm{A}}_{11} + \bar{\mathbf{C}}_{\mathrm{s}2}\bar{\mathbf{A}}_{21}, \\ \mathbf{K}_2 &= \bar{\mathbf{H}}_2\bar{\mathbf{F}}_2 \quad \text{and} \quad \mathbf{Q}_2 = \bar{\mathbf{H}}_1\bar{\mathbf{G}}_2 + \bar{\mathbf{C}}_{\mathrm{s}1}\bar{\mathrm{A}}_{12} + \bar{\mathbf{C}}_{\mathrm{s}2}\bar{\mathbf{A}}_{22} \end{aligned} \tag{80}$$

Considering the relation (44) between the transformed state variables $\mathbf{y}$ and the vectors $\boldsymbol{\eta}_{\mathrm{a}}$ and $\boldsymbol{\eta}_{\mathrm{u}}$, the time derivative of V can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) = \mathbf{s}^{\mathrm{T}}\{\mathbf{K}_1\boldsymbol{\varphi}_1 + \mathbf{K}_2\boldsymbol{\varphi}_2 + \mathbf{Q}_{\mathrm{a}}\boldsymbol{\eta}_{\mathrm{a}} + \mathbf{Q}_{\mathrm{u}}\boldsymbol{\eta}_{\mathrm{u}}\} + \mathbf{s}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}} + \mathbf{s}^{\mathrm{T}}\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{B}}_{\mathrm{e}}\mathbf{u} \tag{81}$$

where

$$\mathbf{Q}_{\mathrm{a}} = \mathbf{Q}_1\mathbf{P}_{12} + \mathbf{Q}_2\mathbf{P}_{22} \quad \text{and} \quad \mathbf{Q}_{\mathrm{u}} = \mathbf{Q}_1\mathbf{P}_{11} \tag{82}$$

The active controller $\mathbf{u}$ is designed according to the following structure:

$$\mathbf{u} = -[\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{B}}_{\mathrm{e}}]^{\ddagger}[\mathrm{K}_{\mathrm{s}} + \boldsymbol{\Delta}\mathbf{C}_{\mathrm{e}}]\boldsymbol{\eta}_{\mathrm{e}} \tag{83}$$

where the matrices $[\bar{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{B}}_{\mathrm{e}}]^{\ddagger}$ and $\mathbf{C}_{\mathrm{e}}$ and the vector $\boldsymbol{\eta}_{\mathrm{e}}$ were defined in equations (26), (73) and (74), respectively. The matrix $\mathbf{K}_{\mathrm{s}}$ is given by

$$\mathbf{K}_{\mathrm{s}} = [\mathbf{K}_1 \;\; \mathbf{K}_2 \;\; \mathbf{Q}_{\mathrm{a}}] \tag{84}$$

and $\boldsymbol{\Delta} = \mathrm{diag}(\delta_i)$ is a matrix of design parameters, with $\delta_i > 0$ for $i = 1, 2, \ldots, m_{\mathrm{s}}$.

Substituting equation (83) into equation (81), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) = \mathbf{s}^{\mathrm{T}}\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}} + \mathbf{s}^{\mathrm{T}}\mathbf{Q}_{\mathrm{u}}\boldsymbol{\eta}_{\mathrm{u}} - \mathbf{s}^{\mathrm{T}}\boldsymbol{\Delta}\mathbf{s} \tag{85}$$

We note that the time derivative of V is affected by the unmeasured variables $\boldsymbol{\eta}_{\mathrm{u}}$ and the external excitation $\ddot{x}_{\mathrm{g}}$ acting as unmeasured disturbances.

Considering the first term of the right-hand side of equation (85), one can write

$$\mathbf{s}^{\mathrm{T}}\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}} \leqslant |\mathbf{s}^{\mathrm{T}}\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}|\ddot{x}_{\mathrm{g}}^{\max} \tag{86}$$

where $\ddot{x}_{\mathrm{g}}^{\max}$ is a bounding value for the ground acceleration to which the structure is likely to be subjected, i.e. $|\ddot{x}_{\mathrm{g}}| \leqslant \ddot{x}_{\mathrm{g}}^{\max}$. This value may be chosen as the maximum ground acceleration value that can be expected at the site.

Additionally, taking into account that

$$|\mathbf{s}^{\mathrm{T}}\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}| \leqslant \tfrac{1}{2}(\|\mathbf{s}\|_2^2 + \|\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\|_2^2) \tag{87}$$

it follows that

$$\mathbf{s}^{\mathrm{T}}\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\ddot{x}_{\mathrm{g}} \leqslant \tfrac{1}{2}\ddot{x}_{\mathrm{g}}^{\max}(\|\mathbf{s}\|_2^2 + \alpha^2) \tag{88}$$

where the parameter α has been defined as $\alpha = \|\overline{\mathbf{C}}_{\mathrm{e}}\bar{\mathbf{e}}_{\mathrm{e}}\|_2$.

A similar expression can be obtained for the term involving the unmeasured variables in equation (85). If $\eta_{\max}$ is a bounding value for the entries of the vector $\boldsymbol{\eta}_{\mathrm{u}}$ such that

$$\|\boldsymbol{\eta}_{\mathrm{u}}\|_2^2 \leqslant n_{\mathrm{u}}\eta_{\max}^2 \tag{89}$$

then one can write

$$\mathbf{s}^{\mathrm{T}}\mathbf{Q}_{\mathrm{u}}\boldsymbol{\eta}_{\mathrm{u}} \leqslant \tfrac{1}{2}\eta_{\mathrm{u}}\eta_{\max}^2(\|\mathbf{s}\|_2^2 + \beta^2) \tag{90}$$

in which the parameter β is defined as $\beta = \|\mathbf{Q}_{\mathrm{u}}\|_2$. For example, in the case of a shear building with a velocity feedback scheme, the vector $\boldsymbol{\eta}_{\mathrm{u}}$ is composed of interstorey drifts for which a practical limiting value can be easily assumed. A better estimate for $\|\boldsymbol{\eta}_{\mathrm{u}}\|_2^2$ can also be obtained by a simple response spectrum analysis for a set of design ground response spectra for a maximum ground acceleration level of $\ddot{x}_{\mathrm{g}}^{\max}$.

Therefore, by considering equations (88) and (90) and the fact that

$$\mathbf{s}^{\mathrm{T}}\boldsymbol{\Delta}\mathbf{s} \geqslant \delta_{\min}\|\mathbf{s}\|_2^2 \tag{91}$$

where $\delta_{\min}$ denotes the smallest diagonal entry of $\boldsymbol{\Delta}$, it can be shown that the time derivative of the Lyapunov function satisfies the following inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}(V) \leqslant \tfrac{1}{2}(\ddot{x}_{\mathrm{g}}^{\max} + n_{\mathrm{u}}\eta_{\max}^2 - 2\delta_{\min})\|\mathbf{s}\|_2^2 + \tfrac{1}{2}(\ddot{x}_{\mathrm{g}}^{\max}\alpha^2 + n_{\mathrm{u}}\eta_{\max}^2\beta^2) \tag{92}$$

Depending upon the choice of the design parameters in the matrix $\boldsymbol{\Delta}$, which in turn defines $\delta_{\min}$, and provided that $\delta_{\min} > \tfrac{1}{2}(\ddot{x}_{\mathrm{g}}^{\max} + n_{\mathrm{u}}\eta_{\max}^2)$, the last inequality establishes a region defined by

$$\|\mathbf{s}\|_2^2 > \frac{\ddot{x}_{\mathrm{g}}^{\max}\alpha^2 + n_{\mathrm{u}}\eta_{\max}^2\beta^2}{2\delta_{\min} - \ddot{x}_{\mathrm{g}}^{\max} - n_{\mathrm{u}}\eta_{\max}^2} \tag{93}$$

where attraction to the sliding surface is guaranteed. Note that in using a continuous sliding-mode controller based on partial information, the control actions cannot guarantee the global asymptotic stability of the points $\mathbf{s} = \mathbf{0}$. However, the design assures that all trajectories will ultimately lie within a bounded neighbourhood of that point.[19–20] Notice that the inequality (93) is a conservative bound and in practice the system will stay in a region closer to the surface $\mathbf{s} = \mathbf{0}$ than the region defined by equation (93).

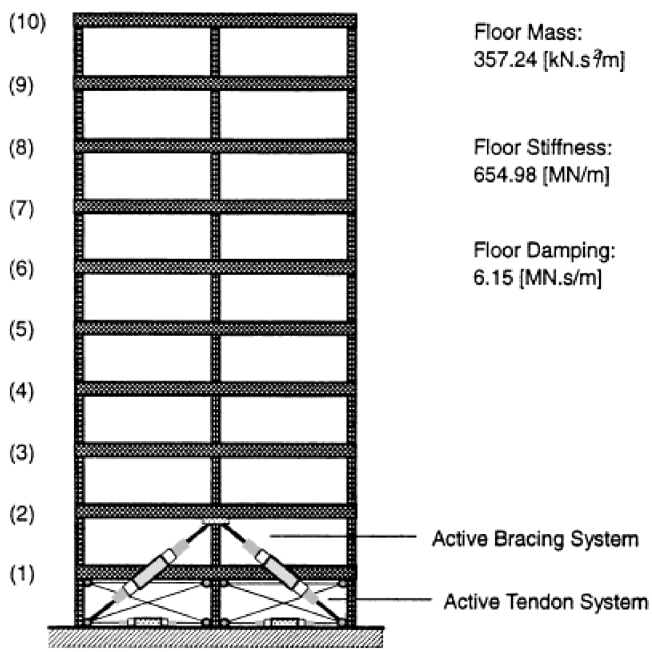
6. NUMERICAL RESULTS

To investigate the performance of the proposed control strategies, numerical simulations have been carried out using a 10-storey shear building model. The structure, shown in Figure 1, represents a typical medium size multi-storey building the floor weights of which correspond to about 400 square meter of floor area on each floor. Each storey has the same mass, stiffness and dashpot parameters, which are indicated in Figure 1. Also shown in the figure are the natural frequencies of this example structure. The damping matrix provided a modal damping ratio of 3 per cent of the critical in the fundamental mode. The active control system considered here consists of two sets of active tendons or bracings installed in the first two storeys which can operate independently. As indicated in the figure, they are assumed to act between the foundation and the first two floor masses.

To evaluate the performance of the proposed control systems for different seismic inputs, four recorded ground motion time histories (El Centro, 1941; San Fernando, 1971; Loma Prieta, 1989; and Kern County—Hollywood site, 1952) have been considered. The time histories for El Centro, San Fernando and Loma Prieta were normalized to a maximum ground acceleration level of 0·3g. The Hollywood record, on the other hand, was normalized to a maximum level of 0·2g, simply to limit the response values of the system. Because of the special frequency characteristics of this motion vis-à-vis the structure's natural frequencies, this ground motion induces relatively large deformations when amplified to higher acceleration levels. Actual maximum ground acceleration for this motion was very low (0·06g).

Full-state feedback

To compare the performance and influence of the number of control devices used, numerical results have been obtained for two different actuation cases. In the first case only one actuator applying force at the first-floor level is used. In the second case, two actuators applying forces independently at the first- and second-floor levels are used. For both cases, the sliding surface is designed using the static relation $\mathbf{q}_1 = \mathbf{y}_1$ in



	Natural Freqs. [rad/sec]
1	6.40
2	19.06
3	31.29
4	42.82
5	53.39
6	62.78
7	70.76
8	77.16
9	81.83
10	84.68

Figure 1. 10-storey shear building model used in the numerical simulations

place of the first auxiliary system defined in equations (17) and (33). This implies that the matrices in equations (17) and (33) associated with this auxiliary system are simply:

$$\bar{\mathbf{F}}_1 = \mathbf{0}, \qquad \bar{\mathbf{G}}_1 = \mathbf{0}, \qquad \bar{\mathbf{D}}_1 = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{E}}_1 = \mathbf{I}_{n_r} \tag{94}$$

To specify the second auxiliary system, penalizing the participation of the variables $\mathbf{y}_2$ in the performance index, the following matrices were chosen:

$$\hat{\mathbf{F}}_2 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta_0\omega_0 \end{bmatrix}, \qquad \hat{\mathbf{G}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \hat{\mathbf{D}}_2 = [0 \;\; \beta_0] \quad \text{and} \quad \hat{\mathbf{E}}_2 = [1] \tag{95}$$

in which ω_0, ζ_0 and β_0 are some design parameters. As will be shown later, the parameters ω_0, ζ_0 and β_0 provide flexibility in adjusting the control efforts conveniently to achieve a desired response reduction. These matrices are used to define the auxiliary system matrices for the two actuation cases mentioned above. For the case of only one actuator on the first floor ($m_c = 1$), the number of sliding constraints is $m_s = 1$ and the matrices associated with the second auxiliary system in equations (17) and (33) are defined as

$$\bar{\mathbf{F}}_2 = \hat{\mathbf{F}}_2, \qquad \bar{\mathbf{G}}_2 = \hat{\mathbf{G}}_2, \qquad \bar{\mathbf{D}}_2 = \hat{\mathbf{D}}_2 \quad \text{and} \quad \bar{\mathbf{E}}_2 = \hat{\mathbf{E}}_2 \tag{96}$$

For the case with two actuation devices ($m_c = 2$), the number of sliding constraints is selected as $m_s = 2$. One could also choose $m_s = 1$, a case with control redundancy. For $m_s = 2$, the matrices associated with the second auxiliary system in equations (17) and (33) are defined as

$$\bar{\mathbf{F}}_2 = \begin{bmatrix} \hat{\mathbf{F}}_2 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{F}}_2 \end{bmatrix}, \qquad \bar{\mathbf{G}}_2 = \begin{bmatrix} \hat{\mathbf{G}}_2 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{G}}_2 \end{bmatrix}, \qquad \bar{\mathbf{D}}_2 = \begin{bmatrix} \hat{\mathbf{D}}_2 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{D}}_2 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{E}}_2 = \begin{bmatrix} \hat{\mathbf{E}}_2 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{E}}_2 \end{bmatrix} \tag{97}$$

For both actuation cases, the weighting matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ in equation (32) are chosen as identity matrices with appropriate dimensions.

Figure 2 shows the time histories of the controlled and uncontrolled top-floor displacement and first-storey shear force for the case $m_c = 2$, when the structure is subjected to El Centro ground acceleration record. The sliding surface design parameters are selected as $\omega_0 = 8{\cdot}00$, $\zeta_0 = 0{\cdot}70$ and $\beta_0 = 85{\cdot}00$. The control actions are governed by equation (83) with the matrix $\boldsymbol{\Delta}$ selected as $\boldsymbol{\Delta} = \delta\mathbf{I}_{m_s}$ in which $\delta = 30{\cdot}00$. In

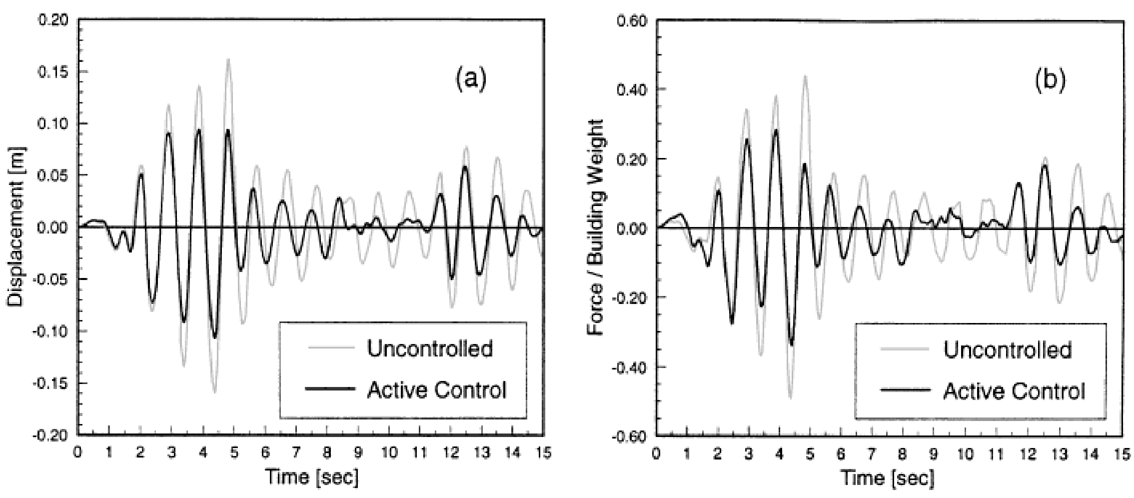


Figure 2. Uncontrolled and controlled responses for El Centro ground excitation: (a) top-floor displacement and (b) first-storey shear force. Full-state feedback

particular, the maximum displacement is reduced to 66 per cent of its maximum uncontrolled value. The effectiveness of the control system in reducing the absolute floor acceleration response and the corresponding floor response spectra is shown in Figure 3; Figure 3(a) shows the time histories of the top-floor-acceleration responses and Figure 3(b) shows the corresponding response spectra, both for the controlled and uncontrolled responses. Both the maximum floor acceleration and the peak floor-response-spectrum values are reduced to about 64 per cent of their uncontrolled response values. The controlled floor response spectrum curve also shows the effectiveness of the control scheme in reducing the response over the entire frequency range.

The time histories of the control forces required at the two floor levels to achieve the results presented in the previous paragraph are shown in Figure 4. It is seen that the proposed control scheme naturally allocates more relevance to the actuation device acting on the higher floor level. This location is obviously more

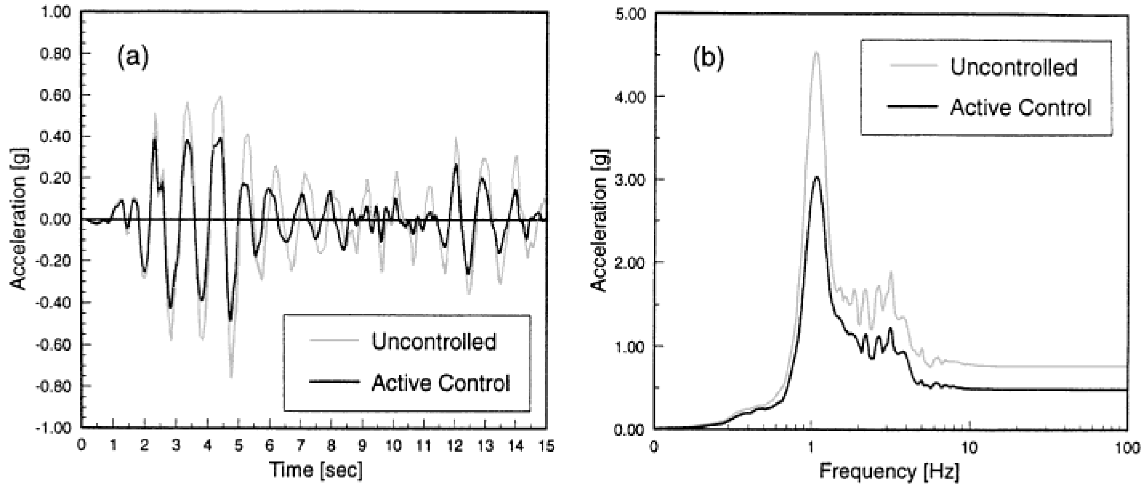


Figure 3. Uncontrolled and controlled responses for El Centro ground excitation: (a) top-floor absolute acceleration and (b) top-floor-response spectrum (3 per cent damping). Full-state feedback

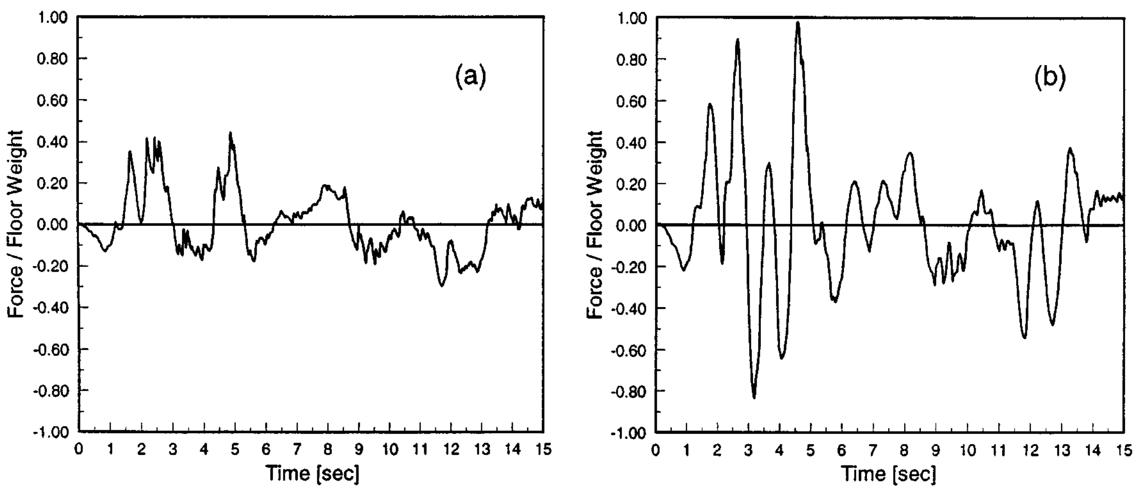


Figure 4. Control force time histories for El Centro ground excitation: (a) first-storey and (b) second-storey devices. Full-state feedback

effective for the control force application to modify the dynamic behaviour of a building system whose response is dominated by the first vibration mode. The maximum values of the control forces required at the first- and second-floor levels are, respectively, 0·45 and 0·98 times the respective floor weights.

For design purposes, it is desirable to assess the maximum control requirements that may be necessary to achieve a pre-set level of reduction in the response. Here this can be conveniently done by changing the parameter ω_0 used for defining the sliding surface. Figure 5(a) shows the levels of response reduction obtained as a function of the parameter ω_0 in terms of response reduction factors. The response reduction factor is defined as the ratio of the controlled to uncontrolled peak values for a given response quantity. The corresponding levels of the maximum control forces required at the two floor levels, as a function of the parameter ω_0, are shown in Figure 5(b). The response reduction curves are monotonic, except for the shear forces in the first two storeys. Since the shear forces in these two storeys are directly affected by the applied control forces, they increase when the control forces are increased beyond a certain level. It is noted that the parameter ω_0 can be used to obtain a family of controller designs for which a decrease in the response is associated with an increase in the control force. That is, the introduction of auxiliary systems in the definition of the sliding surface has made it possible to obtain a proper sliding surface conveniently such that an increase in the control effort translates into an increased reduction in the response. In a traditional sliding-mode-control approach without these compensators or auxiliary systems, obtaining a proper sliding surface is not so straightforward as some choices of parameters could lead to a situation where increased control effort may not result in further reduction of the response. This problem with the classical sliding mode control approach was pointed out earlier by the authors.[10]

Next, the effectiveness of the control system to reduce the responses at various levels of the building when the building is subjected to different earthquakes is investigated. The controlled response obtained with two actuators at floors 1 and 2 ($m_c = 2$) is also compared with the response obtained with a single actuator acting at floor 1 ($m_c = 1$). For a meaningful comparison of the results of the two cases, the control system for the single actuator case is designed such that it produces the same response reduction for the top floor displacement as the reduction caused by two actuators when the structure is subjected to El Centro motion. This can be achieved if the sliding surface design parameters for the single actuator case are chosen as $\omega_0 = 8{\cdot}00$, $\zeta_0 = 0{\cdot}75$ and $\beta_0 = 42{\cdot}00$, and the control parameter δ is chosen as $\delta = 80{\cdot}00$. Table I shows, for each earthquake, the relative displacement response reduction factors of all floor levels. It is noted that the

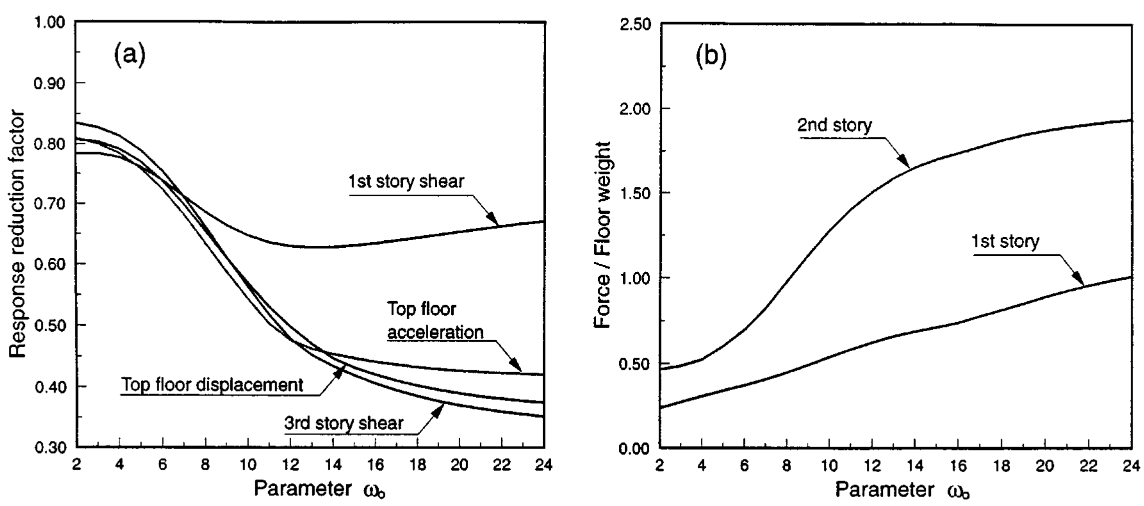


Figure 5. Building responses and control requirements as a function of the parameter ω_0 for El Centro ground excitation. Full-state feedback

Table I. Maximum relative displacements

	El Centro			Hollywood			San Fernando			Loma Prieta		
		Response ratio			Response ratio			Response ratio			Response ratio	
Floor	Response (cm)	$m_c=1$	$m_c=2$	Response (cm)	$m_c=1$	$m_c=2$	Response (cm)	$m_c=1$	$m_c=2$	Response (cm)	$m_c=1$	$m_c=2$
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]
10	**16·25**	0·66	0·66	**15·94**	0·69	0·66	**8·92**	0·73	0·73	**6·17**	0·92	0·89
9	**15·87**	0·66	0·66	**15·58**	0·69	0·66	**8·68**	0·73	0·72	**5·96**	0·93	0·90
8	**15·14**	0·67	0·66	**14·87**	0·68	0·66	**8·21**	0·73	0·72	**5·74**	0·90	0·88
7	**14·13**	0·67	0·67	**13·81**	0·68	0·66	**7·53**	0·72	0·72	**5·23**	0·86	0·83
6	**12·90**	0·68	0·67	**12·44**	0·68	0·66	**6·69**	0·72	0·71	**5·23**	0·80	0·77
5	**11·36**	0·68	0·67	**10·78**	0·68	0·66	**5·71**	0·71	0·71	**4·78**	0·79	0·75
4	**9·54**	0·69	0·67	**8·88**	0·68	0·65	**4·70**	0·73	0·70	**4·14**	0·78	0·73
3	**7·45**	0·70	0·67	**6·78**	0·68	0·65	**3·68**	0·74	0·70	**3·29**	0·76	0·69
2	**5·13**	0·72	0·68	**4·56**	0·68	0·64	**2·56**	0·76	0·70	**2·27**	0·72	0·66
1	**2·63**	0·82	0·69	**2·28**	0·81	0·67	**1·32**	0·83	0·69	**1·15**	0·77	0·68

Table II. Maximum control requirements

	Maximum control forces		
		$m_c = 2$	
	$m_c = 1$ 1st Storey (Floor weight)	1st Storey (Floor weight)	2nd Storey (Floor weight)
	[1]	[2]	[3]
El Centro	2·15	0·45	0·98
Hollywood	2·14	0·39	0·99
San Fernando	1·33	0·33	0·66
Loma Prieta	0·97	0·28	0·49

control system with two active devices performs in general better than the system with only one actuation device as the controlled responses are noted to be somewhat smaller for lower storeys in the former case. Also, from the results for different excitations in the table it is observed that the effectiveness of the control systems does depend on the ground excitation. The maximum control force values required to achieve these reductions are shown in Table II. For all cases of ground excitation, it is noted that the control system with a single actuator on the first storey level demands a much higher level of control actions to produce response reductions comparable to those for the two actuators.

Output feedback

The previous set of results were obtained for the full-state-feedback case. That is, it was assumed that measurements are available for all physical states. Next, we present the results obtained when only a limited set of measurements are available. In this part, only the two-actuator case ($m_c = 2$) is considered; i.e. the building is equipped with two actuation devices acting on the first- and second-floor levels. Two different cases, utilizing different output information and different auxiliary systems to define their sliding surfaces are

considered to demonstrate the generality of the formulation presented in this paper. In one of the two cases, the control redundancy (i.e. $m_c > m_s$) is also considered.

In the first case, it is assumed that the set of measured quantities consist of the interstorey drifts corresponding to the first two storeys where the actuation occurs, and the interstorey drift velocities for the 1st, 2nd, 3rd, 6th, 8th and 10th storeys ($n_a = 8$). The number of sliding constraints for this case is selected as $m_s = 2$; i.e. there is no control redundancy. The corresponding sliding surface is designed for the auxiliary system matrices defined by equations (94) and (97). That is, only the second auxiliary system with $\mathbf{y}_2$ variables is used. The sliding surface parameters for this case are chosen to be $\omega_0 = 8{\cdot}00$, $\zeta_0 = 0{\cdot}70$, $\beta_0 = 158{\cdot}50$, and the control parameter is chosen as $\delta = 150{\cdot}00$. The parameters are selected such that a response reduction factor of 0·66 for the top floor displacement under the El Centro excitation is achieved.

In the second case, it is assumed that the measured quantities are the 1st and 2nd interstorey drifts and all drift velocities ($n_a = 12$). That is more velocity measurements are available for this case than the first case. The number of sliding constraints for this case is selected as $m_s = 1$. Since $m_c > m_s$, the system has control redundancy. Another difference in this case is that a compensator is now used only with the variables $\mathbf{y}_1$ and not with $\mathbf{y}_2$. That is, the second auxiliary system is a static one with $\mathbf{q}_2 = \mathbf{y}_2$ and $n_{s_2} = 0$, and therefore

$$\bar{\mathbf{F}}_2 = \mathbf{0}, \qquad \bar{\mathbf{G}}_2 = \mathbf{0}, \qquad \bar{\mathbf{D}}_2 = \mathbf{0} \quad \text{and} \quad \bar{\mathbf{E}}_2 = \mathbf{I}_{m_s} \tag{98}$$

To penalize the participation of the variables $\mathbf{y}_1$ in the performance index, the following matrices associated with the auxiliary system (51) are used:

$$\bar{\mathbf{F}}_1 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta_0\omega_0 \end{bmatrix}, \qquad \hat{\mathbf{G}}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 24 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \end{bmatrix} \tag{99}$$

$$\bar{\mathbf{D}}_1 = [0 \;\; 1] \quad \text{and} \quad \hat{\mathbf{E}}_1 = [0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;0\;\;1] \tag{100}$$

For this case, the weighting matrices in equation (32) are chosen as $\mathbf{Q}_1 = q_1$, where q_1 is a scalar parameter, and $\mathbf{Q}_2 = \mathbf{I}_{m_s}$. For a meaningful comparison of the results obtained for this case with those of the other cases, the control system design parameters are again selected such that the maximum top floor displacement under El Centro excitation is reduced by a factor of 0·66. This is achieved when the sliding surface parameters are $\omega_0 = 6{\cdot}00$, $\zeta_0 = 0{\cdot}50$ and $q_1 = 10{\cdot}50$, and the control parameter $\delta = 150{\cdot}00$.

The next set of figures and tables provide the numerical results obtained for these two output-feedback cases. Since the general characteristics of the response results are similar to those presented earlier, only a few figures are shown. For example, both for controlled and uncontrolled cases, the time histories of the top floor displacement and the control force in first floor device are shown in Figures 6(a) and 6(b), respectively. These results are for case 1 with fewer measurements. The results for the case 2 were also similar qualitatively. The results in Figure 7 are for the floor acceleration response and response spectra, but for the case 2. The results for case 1 were also similar qualitatively.

The results in the next set of three tables compare the output-feedback and full-state-feedback results obtained for the four different earthquake inputs. Table III shows the relative displacement response reduction factors, whereas Table IV shows the floor acceleration response reduction factors. Since the sliding surface and control parameters for each case were adjusted to obtain the same level of top-floor-displacement reduction, response reduction factors in Table III for different control schemes are in about the same range with some variations. The effectiveness of the control schemes in reducing their peak displacement response, however, can vary with the input. For the results presented in Table III, all control schemes in general seem to be less effective for the San Fernando and Loma Prieta earthquake motions. Acceleration results in Table IV are qualitatively similar to those in the previous table. For the acceleration responses, the output feedback case 2 seems to be somewhat better than others, especially for the Loma Prieta input.

Table V compares the level of control effort required in the three different control schemes to achieve the response reduction depicted in Tables III and IV. It is seen from the values in the table that to achieve about the same level of reduction, the full-state feedback requires the lowest amount of the control force. Also the

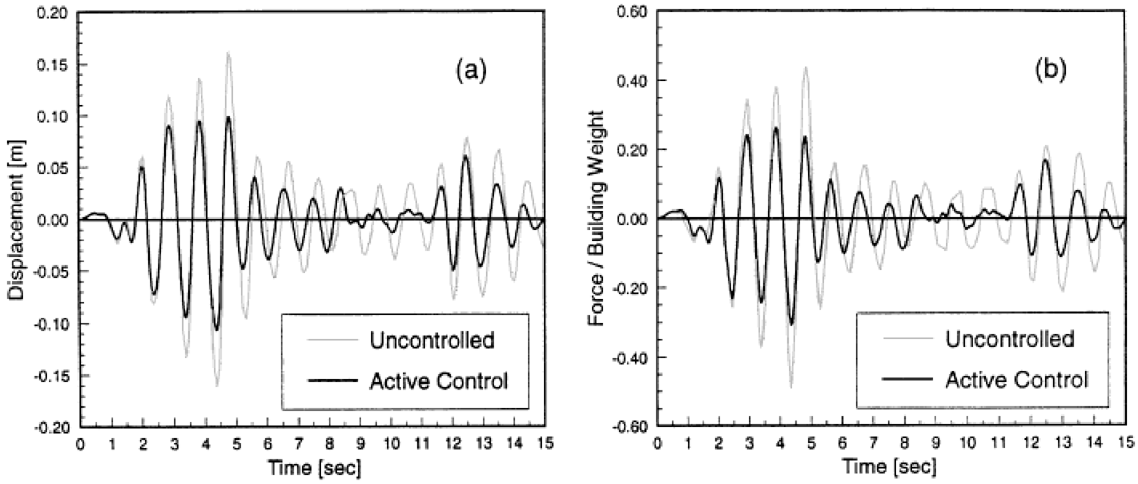


Figure 6. Uncontrolled and controlled responses for El Centro ground excitation: (a) top-floor displacement and (b) first-storey shear force. Output feedback

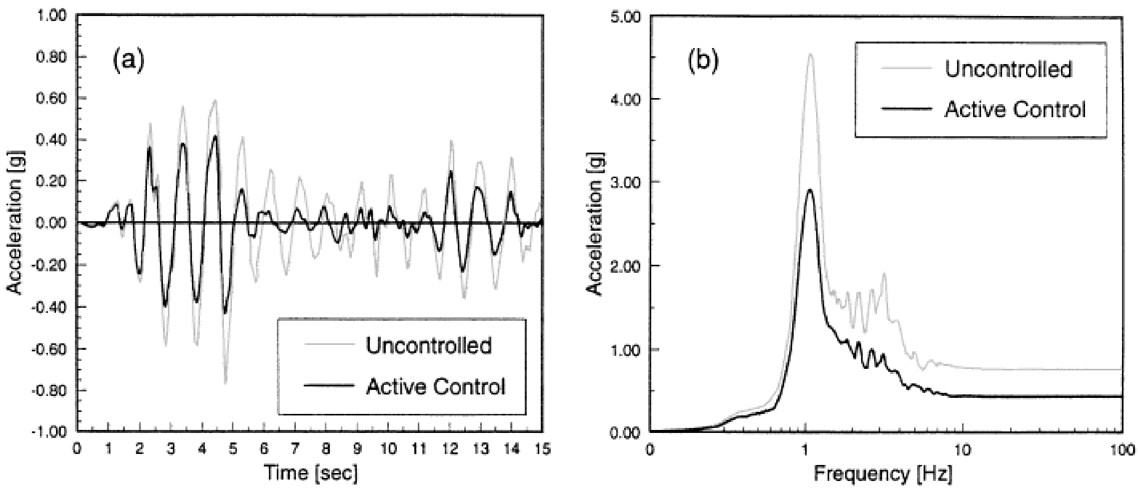


Figure 7. Uncontrolled and controlled responses for El Centro ground excitation: (a) top-floor absolute acceleration and (b) top-floor response spectrum (3 per cent damping). Output feedback

output-feedback case 1 where the number of available states were 8, in general, is seen to require larger maximum forces compared to the case 2 where the number of available states were 12. Thus, the lesser the number of states available for feedback, the larger the level of control effort required to achieve a given level of reduction in the response. The sliding-mode-control formulation developed here confirms this naturally expected conclusion.

To show that $\mathbf{y}_1$ weighting with output feedback is also equally convenient to achieve a proper sliding surface where an increased control effort leads to an increased reduction in the response, the results in Figure 8 are reported. The monotonic increase in response reduction factors achieved in various storey shears, top-floor displacement and top-floor acceleration is consistent with the monotonic decrease in the control forces at the first- and second-floor levels. The first-storey shear reduction is not monotonic because this response quantity is directly affected by the control force.

Table III. Maximum relative displacements

	El Centro				Hollywood				San Fernando				Loma Prieta			
		Response ratio				Response ratio				Response ratio				Response ratio		
Floor	Response (cm)	Full-state feedback	(a) Output feedback	(b) Output feedback	Response (cm)	Full-state feedback	(a) Output feedback	(b) Output feedback	Response (cm)	Full-state feedback	(a) Output feedback	(b) Output feedback	Response (cm)	Full-state feedback	(a) Output feedback	(b) Output feedback
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]
10	**16·25**	0·66	0·66	0·66	**15·94**	0·66	0·67	0·64	**8·92**	0·73	0·73	0·71	**6·17**	0·89	0·88	0·85
9	**15·87**	0·66	0·66	0·66	**15·58**	0·66	0·67	0·64	**8·68**	0·72	0·73	0·71	**5·96**	0·90	0·89	0·85
8	**15·14**	0·66	0·66	0·66	**14·87**	0·66	0·66	0·64	**8·21**	0·72	0·73	0·70	**5·74**	0·88	0·87	0·83
7	**14·13**	0·67	0·67	0·66	**13·81**	0·66	0·66	0·64	**7·53**	0·72	0·72	0·70	**5·53**	0·83	0·82	0·79
6	**12·90**	0·67	0·66	0·66	**12·44**	0·66	0·66	0·64	**6·69**	0·71	0·72	0·70	**5·23**	0·77	0·79	0·73
5	**11·36**	0·67	0·66	0·66	**10·78**	0·66	0·66	0·64	**5·71**	0·71	0·71	0·72	**4·78**	0·75	0·76	0·69
4	**9·54**	0·67	0·66	0·66	**8·88**	0·65	0·65	0·65	**4·70**	0·70	0·70	0·74	**4·14**	0·73	0·74	0·66
3	**7·45**	0·67	0·66	0·66	**6·78**	0·65	0·64	0·67	**3·68**	0·70	0·70	0·74	**3·29**	0·69	0·70	0·64
2	**5·13**	0·68	0·65	0·67	**4·56**	0·64	0·64	0·71	**2·56**	0·70	0·69	0·78	**2·27**	0·66	0·66	0·64
1	**2·63**	0·69	0·63	0·68	**2·28**	0·67	0·63	0·74	**1·32**	0·69	0·65	0·79	**1·15**	0·68	0·67	0·63

Table IV. Maximum absolute accelerations

	El Centro				Hollywood				San Fernando				Loma Prieta			
		Response ratio				Response ratio				Response ratio				Response ratio		
Floor	Response [g]	Full-state feedback	(a) Output feedback	(b) Output feedback	Response [g]	Full-state feedback	(a) Output feedback	(b) Output feedback	Response [g]	Full-state feedback	(a) Output feedback	(b) Output feedback	Response [g]	Full-state feedback	(a) Output feedback	(b) Output feedback
[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]	[16]	[17]
10	**0·76**	0·64	0·66	0·57	**0·69**	0·67	0·68	0·67	**0·48**	0·78	0·79	0·74	**0·49**	0·71	0·72	0·62
9	**0·72**	0·64	0·67	0·57	**0·67**	0·67	0·68	0·66	**0·44**	0·78	0·79	0·75	**0·42**	0·73	0·74	0·67
8	**0·64**	0·64	0·67	0·62	**0·64**	0·67	0·68	0·66	**0·39**	0·75	0·76	0·70	**0·33**	0·79	0·80	0·74
7	**0·60**	0·64	0·66	0·65	**0·60**	0·66	0·67	0·64	**0·35**	0·72	0·73	0·62	**0·28**	0·80	0·81	0·69
6	**0·58**	0·65	0·66	0·63	**0·55**	0·66	0·67	0·63	**0·35**	0·71	0·72	0·64	**0·36**	0·73	0·75	0·62
5	**0·54**	0·68	0·69	0·65	**0·48**	0·65	0·67	0·63	**0·41**	0·66	0·67	0·54	**0·42**	0·69	0·70	0·57
4	**0·50**	0·68	0·70	0·65	**0·42**	0·65	0·67	0·65	**0·43**	0·63	0·64	0·51	**0·43**	0·67	0·68	0·54
3	**0·44**	0·69	0·71	0·66	**0·36**	0·65	0·66	0·64	**0·37**	0·62	0·63	0·58	**0·37**	0·64	0·66	0·51
2	**0·35**	0·67	0·67	0·75	**0·29**	0·58	0·58	0·60	**0·33**	0·67	0·68	0·62	**0·28**	0·65	0·67	0·61
1	**0·26**	0·85	0·91	0·87	**0·21**	0·74	0·79	0·76	**0·31**	0·80	0·84	0·80	**0·29**	0·80	0·81	0·78

Table V. Maximum control requirements

	Maximum control forces					
	Full-state feedback		(a) Output feedback		(b) Output feedback	
	1st Storey [Floor weight]	2nd Storey [Floor weight]	1st Storey [Floor weight]	2nd Storey [Floor weight]	1st Storey [Floor weight]	2nd Storey [Floor weight]
	[1]	[2]	[3]	[4]	[5]	[6]
El Centro	0·45	0·98	0·59	1·31	0·56	1·12
Hollywood	0·39	0·99	0·40	1·23	0·50	1·00
San Fernando	0·33	0·66	0·39	0·84	0·40	0·80
Loma Prieta	0·28	0·49	0·30	0·58	0·29	0·58

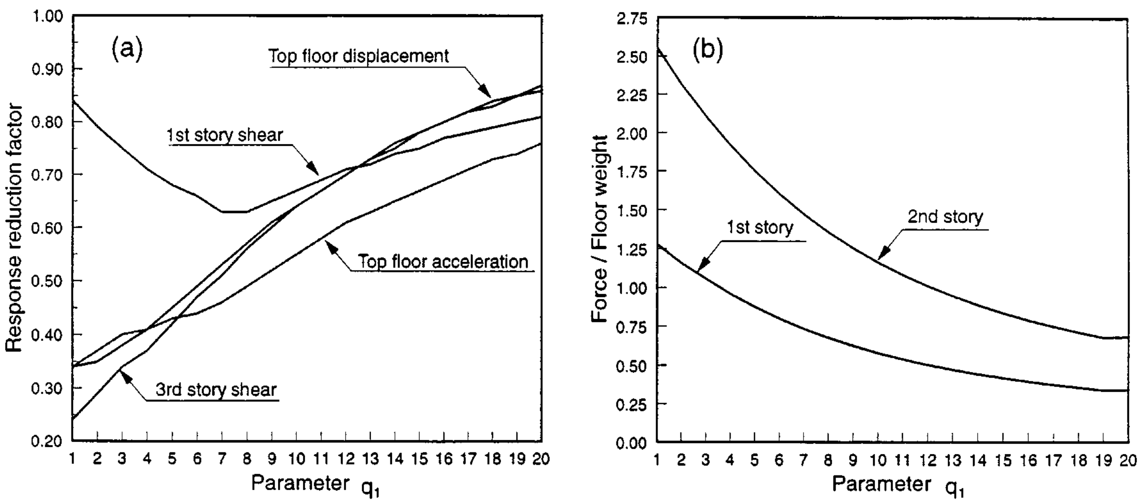


Figure 8. Building responses and control requirements as a function of the parameter q_1 for El Centro ground excitation. Output feedback

7. CONCLUDING REMARKS

The paper presents the development of sliding-mode-control algorithms for the full-state- and output-feedback cases. A systematic approach is presented to achieve the regular form of the equation of motion to effect an uncoupling of sliding motion and control actions. The approach is general enough such that it can be applied to cases where a simple re-ordering of the equations of motion will not achieve these uncoupling objectives. To gain additional flexibility in the design of a better sliding surface, two auxiliary dynamical systems are introduced. The sliding surface is defined in terms of the outputs of these auxiliary systems. These auxiliary systems implicitly introduce a frequency weighting in the cost function used for calculating the sliding constriants. A general formulation is presented to eliminate the contribution of the unmeasured state variables in the output-feedback case. To calculate the sliding surface, this case requires a constrained minimization of the performance index. This requires the solution of three non-linear matrix equations and a new sequential procedure with quick convergence is used for their solution.

Numerical results are presented to show the applicability of the formulation and effectiveness of action control in reducing the deformation and acceleration responses. The structure examined is a realistic

10-storey building. Two independent actuators are used to apply control force through tendons at the first- and second-floor levels. With this actuation arrangements and the level of response reduction achieved, the control force requirements are reasonable. Earlier much higher levels of control forces were reported.[10] This has been possible partially because of a more efficient design of the sliding surface, and also because of the use of the two control tendons instead of only one applied in the earlier study. The effectiveness of the output-feedback-control scheme is also demonstrated. There is only slight degradation in the performance of the control because of unavailability of all the system states, but the system stability is maintained. The scheme presented can successfully eliminate the effect of unmeasured states from the design of the sliding surface and the controller. It is also shown that the introduction of auxiliary systems in the design of sliding surface is effective in achieving a proper sliding surface.

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